A Power Series expansion about $x = a$ with coefficient Sequence C_n is given by $\sum_{n=0}^{\infty} C_n (x-a)^n$.

Considering all constants to be real numbers with x

The two types of series encountered in calculus are taylor and Maclaurin Series.

A Taylor Series expansion of $f(x)$ about $x = a$ is the series

$$f(x) \sim \sum_{n=0}^{\infty} C_n (x-a)^n$$

Where

$$C_n = \frac{f^{(n)}(a)}{n!}$$

Note that $\sim$ sign shows that we are yet to determine when the series may converge to ~~the~~ given function

A Maclaurin Series expansion of $f(x)$ is of $f(x)$ about $x = 0$, or Taylor series expansion

$$f(x) \sim \sum_{n=0}^{\infty} C_n x^n,$$

where

$$C_n = \frac{f^{(n)}(0)}{n!}$$

We note that Maclaurin series are a special case of Taylor series in which the expansion is about $x = 0$.

$y'(x) = x + y(x), \quad y(0) = 1$

Assume we write the solution as the Maclaurin series

$$y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(x)}{n!} x^n$$

$$= y(0) + y'(0)x + \frac{1}{2} y''(0)x^2 + \frac{1}{6} y'''(0)x^3 + \cdots$$

$y(0) = 1,$

$y'(0) = 0 + y(0) = 1$

In order to obtain values of the higher order derivatives at $x=0$, we differentiate the differential equation several times

$$y''(x) = 1 + y'(x)$$

$$y''(0) = 1 + y'(0) = 2$$

$$y'''(x) = y''(x) = 2$$

All other values of the derivatives are the same.

Therefore, $y(x) = 1 + x + 2\left(\frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots\right.$

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This solution can be summed as

(Cdi)

$$y(x) = 2\left(1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \dots\right) - 1 - x$$

$$= 2e^x - x - 1.$$

* Find a general Maclaurin Series Solution to the ODE. $y' - 2xy = 0$

Lets assume that the solution takes the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

The goal is to find the expansion Coefficients $c_n, n = 0, 1, \dots$ Differentiating, we have

$$y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

Inserting the series for $y(x)$ and $y'(x)$ into the differential equation, we have

$$0 = \sum_{n=1}^{\infty} n c_n x^{n-1} - 2x \sum_{n=0}^{\infty} c_n x^n$$

$$= (c_1 + 2c_2 x + 3c_3 x^2 + 4x^3 + \dots)$$

$$= -2x(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots)$$

$$= c_1 + (2c_2 - c_0)x + (3c_3 - 2c_1)x^2 + (4c_4 - 2c_2)x^3 +$$

Equating like powers of x on both sides

$$0 = C_1$$
$$0 = 2C_2 - C_0$$
$$0 = 3C_3 - C_1$$
$$0 = 4C_4 - 2C_2$$

We can solve these sequentially for the coefficient of largest index. $\therefore$ $C_1 = 0$, $C_2 = C_0$, $C_3 = \frac{2}{3}C_1 = 0$, $C_4 = \frac{1}{2}C_2 = \frac{1}{2}C_0$

We note that the odd terms vanish and the even terms survive.

$$y(x) = C_0 + C_1x + C_2x^2 + C_3x^3 + \cdots$$
$$= C_0 + C_0x^2 + \frac{1}{2}C_0x^4 + \cdots$$

Series Solution

For General Solution, Combining all terms with like powers of x.

$$0 = \sum_{n=1}^{\infty} nC_nx^{n-1} - 2x\sum_{n=0}^{\infty} C_nx^n$$

$$= \sum_{n=1}^{\infty} nC_nx^{n-1} - \sum_{n=0}^{\infty} 2C_nx^{n+1}$$

Note that the powers of x in these two sums differ by 2. We can re-index the sums separately so that the powers are the same, say 'K'.

So let $K = n-1$ or $n = K+1$ in the first series

$$\sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{k=0}^{\infty} (k+1) c_{k+1} x^k$$

$$= c_1 + 2c_2 x + 3c_3 x^2 + 4x^3 + \cdots$$

Note that re-indexing has not changed the terms in the series. Similarly, we can let $k = n+1$ or $n = k-1$, in the second series to find

$$\sum_{n=0}^{\infty} 2c_n x^{n+1} = \sum_{k=1}^{\infty} 2c_{k-1} x^k$$

$$= 2c_0 + 2c_1 x + 2c_2 x^2 + 2c_3 x^3 + \cdots$$

Combining both series, we have

$$0 = \sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} 2c_n x^{n+1}$$

$$= \sum_{k=0}^{\infty} (k+1) c_{k+1} x^k - \sum_{k=1}^{\infty} 2c_{k-1} x^k$$

$$= c_1 + \sum_{k=1}^{\infty} \left[(k+1) c_{k+1} - 2c_{k-1} \right] x^k$$

Here, we have combined the two series for $k = 1, 2, \cdots$, then $k=0$ term in the first series gives the constant term as shown.

We can now set the coefficients of powers of x equal to zero since there are no terms on the left side of the equation. This gives

$c_1 = 0$ and $(k+1)c_{k+1} - 2c_{k-1}$, $k = 1, 2, \ldots$

This equation is called a recurrence relation. It can be used to find successive coefficients in terms of previous values.

$$c_{k+1} = \frac{2}{k+1} c_{k-1}, \quad k = 1, 2, \ldots$$

Inserting different values of 'k', we have

$k = 1$: $c_2 = \frac{2}{2} c_0 = c_0$, $k = 3$: $c_4 = \frac{2}{4} c_2 = \frac{1}{2} c_0$

$k = 2$ $c_3 = \frac{2}{3} c_1 = 0$ $k = 4$: $c_5 = \frac{2}{5} c_3 = 0$

$k = 5$: $c_6 = \frac{2}{6} c_4 = \frac{1}{3(2)} c_0$.

Thus: $$y(x) = \sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

$$= c_0 + c_0 x^2 + \frac{1}{2!} c_0 x^4 + \frac{1}{3!} c_0 x^6 + \ldots$$

$$= c_0 \left(1 + x^2 + \frac{1}{2!} x^4 + \frac{1}{3!} x^6 + \ldots\right)$$

$$= c_0 \sum_{l=0}^{\infty} \frac{1}{l!} x^{2l}$$

$$= c_0 e^{x^2}$$

Power Series

An expression of the type

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n$$

is called a power series in x around x_0. The point x_0 is called the centre, and a_n's are called the coefficients.

Note, $a_n \in R$ is the coefficient of $(x-x_0)^n$ and that the power series converges for $x = x_0$. So the set

$$S = \{x \in R : \sum_{n=0}^{\infty} a_n (x-x_0)^n \text{ converges}\}$$

is a non-empty. Thus, the set S is an interval in R.

Example

Consider the power series $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

In this case, $x_0 = 0$ is the centre, $a_0 = 0$, and $a_{2n} = 0$ for $n \geq 1$. Also, $a_{2n+1} = \frac{(-1)^n}{(2n+1)}$, $n = 1, 2, \ldots$

Any polynomial $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ is a power series with $x_0 = 0$ as the centre, and the coefficients $a_m = 0$ for $m \geq n+1$.

Properties of Power Series

Consider two Power Series

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n (x-x_0)^n$$

With radius of convergence $R_1 > 0$ and $R_2 > 0$, respect-ively. Let $F(z)$ and $G(z)$ be the functions defined by the power series defined for all $x \in I$

Where $I = (-R \text{ to } x_0, x_0 + R)$ with $R = \min\{R_1, R_2\}$

Note that both power series converge for all $x \in I$

With $F(x)$, $G(x)$ and I as defined above, the properties of power series are thus:

1. EQUALITY of Power Series:

the two power series $F(x)$ and $G(x)$ are equal for all $x \in I$ if and only if

$$a_n = b_n \text{ for all. } n = 0, 1, 2, \ldots$$

In particular, if $\sum_{n=0}^{\infty} a_n (x-x_0)^n = 0$ for all $x \in I$, then

$$a_n = 0 \text{ for all } n = 0, 1, 2, \ldots$$

2. Term by Term Addition

for all $x \in I$, we have

$$F(x) + G(x) = \sum_{n=0}^{\infty} (a_n + b_n)(x - x_0)^n$$

essentially, it says that in the common part of the regions of convergence, the two power series can be added term by term

3. Multiplication of Power Series

for $c_0 = a_0 b_0$, and inductively $c_n = \sum_{j=1}^{n} a_{n-j} b_j$

Then for all $x \in I$, the products of $F(x)$ and $G(x)$ is defined by

$$H(x) = F(x)\, G(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

$H(x)$ is called the Cauchy product of $f(x)$ and $G(x)$

Note that for any $n \geq 0$, the coefficient of x^n in

$$\left(\sum_{j=0}^{\infty} a_j (x - x_0)^j\right) \cdot \left(\sum_{k=0}^{\infty} b_k (x - x_0)^k\right) \text{ is } c_n = \sum_{j=1}^{n} a_{n-j} b_j$$

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4. Term by Term Differentiation

For differentiation of the power series function $F(x)$, we have –

$$\sum_{n=1}^{\infty} n a_n (x-x_0)^n$$

Note that it also has R_1 as the radius of converg–
–ence

Let $0 < r < R_1$, then for all $x \in (-r+x_0, x_0+r)$, we have

$$\frac{d}{dx} F(x) = F'(x) = \sum_{n=1}^{\infty} n a_n (x-x_0)^n$$

Solutions in terms of Power Series

Consider a linear second order equation of the type

$$y'' + a(x)y' + b(x)y = 0 \quad \text{—— (1)}$$

Let a and b be analytic around the point $x_0 = 0$

thus: $$y = \sum_{k=0}^{\infty} C_k x^k \quad \text{—— (2)}$$

Substitute (2) in (1) and by finding values of C_k's

Example

Consider the differential equation

$$y'' + y = 0 \quad — (1)$$

Here, $a(x) \equiv 0$, $b(x) \equiv 1$, which are analytic around $x_0 = 0$

Solution

Let $y = \sum_{n=0}^{\infty} C_n x^n$ — (2)

$y' = \sum_{n=0}^{\infty} n C_n x^{n-1}$ and $y'' = \sum_{n=0}^{\infty} n(n-1) C_n x^{n-2}$

Substitute y, y' and y'' in equation (1), we get

$$\sum_{n=0}^{\infty} n(n-1) C_n x^{n-2} + \sum_{n=0}^{\infty} C_n x^n = 0$$

or equivalently

$$0 = \sum_{n=0}^{\infty} (n+2)(n+1) C_{n+2} x^n + \sum_{n=0}^{\infty} C_n x^n = \sum_{n=0}^{\infty} \{(n+1)(n+2) C_{n+2} + C_n\} x^n$$

Hence for all $n = 0, 1, 2, ---$

$(n+1)(n+2) C_{n+2} + C_n = 0$ or

$$C_{n+2} = -\frac{C_n}{(n+1)(n+2)}$$

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therefore, we have -

$C_2 = \frac{-C_0}{2!}$, $C_3 = -\frac{C_1}{3!}$, $C_4 = (-1)^2 \frac{C_0}{4!}$

$C_4 = (-1)^2 \frac{C_0}{4!}$ $\qquad$ $C_5 = (-1)^2 \frac{C_1}{5!}$

$\vdots \qquad \vdots$

$C_{2n} = (-1)^n \frac{C_0}{(2n)!}$ $\qquad$ $C_{2n+1} = (-1)^n \frac{C_1}{(2n+1)!}$

Here, C_0 and C_1 are arbitrary, so.

$$y = C_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + C_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

ie

$y = C_0 \cos(x) + C_1 \sin(x)$ where C_0 and C_1 can be chosen arbitrary for $C_0 = 1$, and $C_1 = 0$, we get $y = \cos(x)$

Thus; $\cos(x)$ and $y = \sin(x)$ are the solution of the equation (1) $y'' + y = 0$